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Balance laws and energy release rates for cracks in dipolar gradient elasticity

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Abstract

It is the purpose of this work to derive the balance laws (in the Günther–Knowles–Sternberg sense) pertaining to dipolar gradient elasticity. The theory of dipolar gradient (or grade 2) elasticity derives from considerations of microstructure in elastic continua [Mindlin, R.D., 1964. Microstructure in linear elasticity. Arch. Rational Mech. Anal. 16, 51–78] and is appropriate to model materials with periodic structure. According to this theory, the strain–energy density assumes the form of a positive-definite function of the strain (as in classical elasticity) and the gradient of both strain and rotation (additional terms). The balance laws are derived here through a more straightforward procedure than the one usually employed in classical elasticity (i.e. Noether's theorem). Indeed, the pertinent balance laws are obtained through the action of the standard operators of vector calculus (*grad*, *curl* and *div*) on appropriate forms of the Hamiltonian of the system under consideration. These laws are directly related to the energy release rates in the processes of crack translation, rotation and self-similar expansion. Under certain conditions, they are identified with conservation laws and path-independent integrals are obtained.

Keywords: Microstructure; Generalized continuum theories; Gradient elasticity; Dipolar stresses; Balance laws; Conservation laws; Path-independent integrals; Energy release rates

1. Introduction

This paper is the third (and the last one) in a series of papers by the present authors dealing with energy-related considerations in dipolar gradient elasticity. It is our purpose here to derive the *balance laws* (in the Günther–Knowles–Sternberg sense) in dipolar gradient elasticity and also related *conservation laws* and path-independent integrals for cracks. The first paper in this series (Grentzelou and Georgiadis, 2005) established a uniqueness theorem for cracks and pertinent edge conditions by employing energy concepts and theorems. These edge conditions are appropriate bounds for certain fields near the crack edges. The second paper (Georgiadis and Grentzelou, 2006) dealt with all basic energy theorems (the theorem of minimum potential

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energy, the analogue of Hellinger–Reissner principle, the analogues of Castigliano and Engesser theorems, and a reciprocal theorem) and, also, with the derivation of a path-independent integral of the *J*-type for cracks in solids governed by the theory of dipolar gradient elasticity.

Since extensive literature reviews and expositions on dipolar gradient elasticity are already given in our recent papers mentioned above, here we only present a brief introduction to the subject. The theory of dipolar gradient elasticity was introduced by Toupin (1962) and Mindlin (1964) in an effort to model the mechanical behavior of solids with *microstructure*. In the present work the general form of gradient theory (Mindlin, 1964) is adopted. This form assumes that: (i) each material particle has three degrees of freedom, (ii) an augmented form of the Euler–Cauchy principle with a non-vanishing couple traction prevails, and (iii) the strain–energy density depends upon the strain tensor and upon the gradient of both strain and rotation (i.e. the second gradient of the displacement field). As explained in Section 2 below, the general form includes as important special cases the strain-gradient elasticity (form II in Mindlin, 1964) and the couple-stress elasticity (special case of form III in Mindlin, 1964). Finally, we notice that the gradient theory is different from the Cosserat (or micropolar) theory that takes material particles with six independent degrees of freedom (three displacement components and three rotation components, the latter involving rotation of a micro-medium w.r.t. its surrounding medium).

An interesting feature of the theory stems from the assumed dependence of the strain energy on the gradient of strain and/or rotation: the new material constants imply the presence of characteristic lengths in the material behaviour. These lengths can be related with the size of microstructure. In this way, size effects can be incorporated in the stress analysis in a manner that classical theories cannot afford. Continua for which such an analysis can be useful are periodic material structures like those, e.g., of crystal lattices, crystallites of a polycrystal or grains of a granular material.

Besides the fundamental papers by Toupin (1962) and Mindlin (1964), important theoretical contributions in gradient elasticity are contained in the works by Bleustein (1967), Mindlin and Eshel (1968), and Germain (1973). Recent developments and applications of the gradient theory (in elasticity and plasticity) include work by, among others, Fleck et al. (1994), Vardoulakis and Sulem (1995), Vardoulakis and Georgiadis (1997), Begley and Hutchinson (1998), Fleck and Hutchinson (1998), Zhang et al. (1998), Chen et al. (1999), Gao et al. (1999), Shu et al. (1999), Huang et al. (2000), Shi et al. (2000), Amanatidou and Aravas (2002), Georgiadis (2003), Huang et al. (2004), Georgiadis et al. (2004), Lazar and Maugin (2005, 2006), Giannakopoulos et al. (2006), Giannakopoulos and Stamoulis (2007), and Lazar and Kirchner (2007).

Regarding now appropriate length scales for strain gradient theories, as noted by Zhang et al. (1998), although strain gradient effects are associated with geometrically necessary dislocations in plasticity, they may also be important for the *elastic* range in microstructured materials. Generally, theories with strain gradient effects are intended to model situations where the intrinsic material lengths are of the order of 0.1–10 μm (see e.g. Shi et al., 2000). Since the strengthening effects arising from strain gradients become important when these gradients are large enough, these effects will be significant when the material is deformed in *very small* volumes, such as in the immediate vicinity of crack tips, notches, small holes and inclusions, and micrometer indentations. Examples of successful modelling of microstructure and size effects in elastically deformed solids by the dipolar gradient theory include propagation of waves with small wavelengths in layered media (Herrmann and Achenbach, 1968), bending of a polycrystalline aluminum beam (Kakunai et al., 1985), buckling of elastic fibres in composites (Fleck and Shu, 1995), granular materials (Chang et al., 2003), and new gradient-elasticity solutions to the Flamant–Boussinesq and Kelvin problems that predict (in contrast with classical elasticity) continuous and bounded displacements at the points of application of the loads (Georgiadis and Anagnostou, 2007). Also, in wave propagation dealing with electronic-device applications, surface-wave frequencies of the order of GHz are often used and therefore wavelengths of the micron order appear (see e.g. White, 1970; Farnell, 1978). In such situations, dispersion phenomena at high frequencies can only be explained on the basis of strain gradient theories (Georgiadis and Velgaki, 2003; Georgiadis et al., 2004). In addition, the latter studies also provide estimates for a single microstructural parameter (i.e. an additional material parameter to the standard Lamé constants λ and μ) employed in some simple material models (like the couple-stress elasticity and the gradient elasticity of form II in Mindlin's theory), which lie within the context discussed here.

We now focus attention to our specific subject, i.e. balance laws, conservation laws and energy release rates for cracks. This is a subject of Mechanics in Material Space (or Configurational Mechanics or Eshelbian Mechanics) (see e.g. the treatise by Kienzler and Herrmann, 2000). The standard methods for deriving balance and conservation laws in mechanics of continua are: (i) application of Noether's (1918) theorem on invariant variational principles, (ii) application of differential operators, and (iii) the Neutral Action method. The first two methods are applicable only for non-dissipative systems (i.e. systems for which a Lagrangian function does exist). For dissipative systems, the third method has been developed (see e.g. Honein et al., 1991; Kienzler and Herrmann, 2000).

Günther (1962) was the first to apply Noether's theorem in linear elastostatics in the absence of defects and obtained the corresponding conservation laws. A partial result of the latter work was the derivation of an integral obtained independently by Eshelby (1951) and Rice (1968), who both followed different procedures. Eshelby (1951) introduced the concept of the force on an elastic defect. He defined this force as the negative gradient of the total potential energy of the body with respect to the size of the defect and proved that it could be evaluated as an integral surrounding the defect. In the absence of such a defect, he proved that this integral is equal to zero giving thus rise to a conservation law. On the other hand, Rice (1968) identified the energy release rate during crack extension with the path-independent integral, known now as J-integral. Günther (1962) related this integral to translation invariance, and also obtained two additional integrals, which he related to rotation and expansion invariance. Knowles and Sternberg (1972) also applied Noether's theorem obtaining the conservation laws for both linearized and finite elastostatics. Next, Budiansky and Rice (1973) indicated that the two additional integrals obtained by Knowles and Sternberg are associated with cavity rotation and cavity expansion and they named them $\boldsymbol{L}$ and M integrals, respectively. Fletcher (1976) extended the work by Knowles and Sternberg for linear elastodynamics and so did Herrmann (1978) for some specific cases of linear elastodynamics and thermoelasticity.

The underlying idea in applying Noether's theorem is that *invariance* of the action of the *Lagrangian* under certain transformations leads to the complete set of balance laws for the system under consideration. Therefore, these laws must in fact lie latently in the Lagrangian itself and might thus be derived by subjecting the Lagrangian function itself to certain operations. It was Golebiewska-Herrmann (1981) who first indicated this, and later, Eischen and Herrmann (1987), carrying out the necessary steps, produced the three balance laws of linear elastodynamics by submitting the Lagrangian function to the usual operators of the vector calculus, namely *grad*, *curl* and *div*. The latter two act in fact on the product of the Lagrangian function and the coordinates. In the words of Eischen and Herrmann (1987), 'While derivations based on Noether's theorem are admittedly more satisfying from a theoretical standpoint, the attendant mathematical apparatus tends to obscure the relative simplicity of the end results. Therefore, alternative means of deriving balance laws for linear elasticity seems to be a credible objective'. Indeed, having obtained these balance laws, Eischen and Herrmann (1987) proceeded in relating them to the energy release rates for certain crack motions, in the spirit of Budiansky and Rice (1973), trying to attach a physical meaning to them. However, Eischen and Herrmann concluded that a minor discrepancy existed between the two sets of equations which, as they pointed out, could have been avoided had the *Hamiltonian* of the system (i.e. the total energy density) been used instead of the Lagrangian.

It is the latter conclusion that we take here as point of departure in the derivation of the balance laws of dipolar gradient elasticity. The Hamiltonian of the system is subjected to the *grad* operator, and the product of the Hamiltonian and the coordinates is subjected to the *curl* and *div* operators. Thus three balance laws are obtained, the first two of which, under certain conditions, produce conservation laws, i.e. path-independent integrals, analogous to the $\boldsymbol{J}$ and $\boldsymbol{L}$ integrals of classical elasticity. We found that a path-independent integral of the M type does not exist in dipolar gradient elasticity. The relation of these balance laws to energy release rates under certain crack motions is also shown.

Closely related studies to the present study are the ones by Kalpakides and Agiasofitou (2002) on conservation laws in gradient electroelasticity and Lazar and Kirchner (2007) on the energy–momentum tensor, angular momentum tensor and dilatational flux in gradient elasticity. We should also mention that Eshelby (1975) touches upon the energy–momentum tensor for materials governed by dipolar gradient elasticity. In addition, Georgiadis (2003) presented an actual calculation of a 2D integral of the J-type for a crack problem in gradient elasticity, whereas Georgiadis and Grentzelou (2006) provided an elementary direct derivation (i.e.

without resorting to the machinery of balance laws, etc.) of a 2D integral of the J-type in dipolar gradient elasticity.

Finally, as for applications of balance and conservation laws these laws have a broad range of applicability (see e.g. Rice, 1968; Nakamura et al., 1985; Olver, 1984; Kienzler and Herrmann, 2000, 2004; Ma et al., 2006): defect and fracture mechanics, stability of surfaces and interfaces, moving phase transformations, melting and mass-accretion. They are also valuable tools in establishing uniqueness and existence theorems and in the improvement of algorithms in numerical procedures.

2. Basic equations of dipolar gradient elasticity

In this Section, we give a brief account of dipolar gradient (or grade 2) elasticity. For more details we refer the reader to our recent papers (Georgiadis et al., 2004; Grentzelou and Georgiadis, 2005; Georgiadis and Grentzelou, 2006) and to the fundamental papers by Toupin (1962), Mindlin (1964), Bleustein (1967), Mindlin and Eshel (1968), and Germain (1973).

In a continuum characterized by the gradient theory, we assume small displacements, strains and rotations and write the first law of thermodynamics as

$$\rho \dot{U}_0 = \tau_{pq}\dot{\varepsilon}_{pq} + m_{rpq}\partial_r\partial_p\dot{u}_q, \tag{1}$$

with respect to a Cartesian rectangular coordinate system $Ox_1x_2x_3$ (indicial notation and the summation convention are used throughout). Here, ρ is the mass density of the continuum, U_0 is the internal energy per unit mass, u_q is the displacement vector, $\varepsilon_{pq} = (1/2)(\partial_p u_q + \partial_q u_p)$ is the linear strain tensor, $\tau_{pq} = \tau_{qp}$ is the monopolar (or Cauchy in the nomenclature of Mindlin, 1964) stress tensor, $m_{rpq} = m_{prq}$ is the dipolar (or double) stress tensor (a third-rank tensor), $\partial_p(\ ) \equiv \partial(\ )/\partial x_p$, a superposed dot denotes time derivative, and the Latin indices span the range $(1, 2, 3)$.

The dipolar stress tensor follows from the notion of dipolar forces, which are anti-parallel forces acting between the micro-media contained in the continuum with microstructure. As explained by Green and Rivlin (1964) and Jaunzemis (1967), the notion of multipolar forces arises from a series expansion of the mechanical power containing higher-order velocity gradients.

Now, compatible with (1) is the following form of the strain–energy density W stored in the continuum

$$W \equiv W(\varepsilon_{pq}, \kappa_{rpq}), \tag{2}$$

where $\kappa_{rpq} \equiv \partial_r\partial_p u_q = \partial_p\partial_r u_q$ is the second gradient of displacement. The kinematical field $(\varepsilon_{pq}, \kappa_{rpq})$ is assumed to be compatible in the sense that the relations $e_{ljp}e_{mqr}\partial_j\partial_r\ \varepsilon_{pq} = 0$ and $e_{ljr}\partial_j\kappa_{rpq} = 0$ (with e_{ljp} being the Levi-Civita alternating symbol) are satisfied (cf. Mindlin, 1964). In what follows, we assume the existence of a *positive definite* function $W(\varepsilon_{pq}, \kappa_{rpq})$. Also, the form in (2) allows not only for a linear constitutive behaviour of the material but also for a non-linear one. From the previous definitions of the kinematical variables, the symmetry relations $\varepsilon_{pq} = \varepsilon_{qp}$ and $\kappa_{rpq} = \kappa_{prq}$ are obvious. Simpler versions of the general form can be derived by identifying κ_{rpq} with either the strain gradient (strain-gradient theory: $\kappa_{rpq} = \partial_r\varepsilon_{pq}$ – form II in Mindlin, 1964) or the rotation gradient (couple-stress theory: $\kappa_{rpq} = \partial_r\omega_{pq}$ – special case of form III in Mindlin, 1964, where $\omega_{pq} = (1/2)(\partial_p u_q - \partial_q u_p)$ is the rotation tensor, following the property $\omega_{pq} = -\omega_{qp}$). Nevertheless, we deal here with the general case by taking the gradient of the *entire* displacement-gradient field.

Further, stresses can be defined in the standard variational manner

$$\tau_{pq} \equiv \frac{\partial W}{\partial \varepsilon_{pq}}, \quad m_{rpq} \equiv \frac{\partial W}{\partial \kappa_{rpq}}, \tag{3a, b}$$

so that $W(\varepsilon_{pq}, \kappa_{rpq}) \equiv (\int_0^{\varepsilon_{pq}} \tau_{pq}\mathrm{d}\varepsilon_{pq} + \int_0^{\kappa_{rpq}} m_{rpq}\mathrm{d}\kappa_{rpq})$. Note that, since m_{pqr} is not symmetric w.r.t. its last two indices, couple-stresses exist in this form because the couple-stress tensor μ_{rl} is related (see e.g. Jaunzemis, 1967) with the dipolar stress tensor m_{pqr} through the equation $\mu_{rl} = (1/2)e_{lpq}m_{r[pq]}$, where $m_{r[pq]} \equiv (1/2)(m_{rpq} - m_{rqp})$.

Next, we consider the kinetic-energy density. Derivations of the full expression of the kinetic-energy density for generalized continua of gradient type can be found in Mindlin (1964) and Georgiadis et al. (2004). Here,

we choose to neglect the effects of velocity gradients, in which case the kinetic-energy density of the material particles assumes its classical form

$$T = (1/2)\rho \dot{u}_q \dot{u}_q. \tag{4}$$

The reason for neglecting an additional term of the form $I\dot{u}_{q,p}\dot{u}_{q,p}$ (with I being a micro-inertia coefficient) in the RHS of (4) is that this term, in the general inertial case, produces a set of equations (it also enters the equation of balance of momentum) that *violates* the assumption of objective tractions. This situation is explained in the paper by Georgiadis et al. (2004). Omitting the micro-inertia term is also the choice in the basic work of Mindlin and Eshel (1968). Anyway, this term is appreciable only for very high frequencies in dynamical problems. We should mention that in the special time-harmonic steady-state case, the non-objectivity is not encountered. In fact, this allowed Georgiadis et al. (2004) to include the micro-inertia term in wave propagation studies. Finally, further discussions on the issue of non-objectivity of tractions caused by the aforementioned micro-inertia term are provided by Jaunzemis (1967, pp. 233) and Eringen (1968).

Then, the equations of motion and the traction boundary conditions along a smooth boundary can be obtained from energy considerations (Mindlin, 1964; Bleustein, 1967; Mindlin and Eshel, 1968). In accord with (1), (2) and (3a,b), the energy equation postulated is written as

$$\int_V \rho \dot{u}_q \ddot{u}_q \mathrm{d}V + \int_V \rho \dot{U}_0 \mathrm{d}V = \int_V f_q \dot{u}_q \mathrm{d}V + \int_S t_q^{(n)} \dot{u}_q \mathrm{d}S + \int_S T_{qr}^{(n)} \partial_q \dot{u}_r \mathrm{d}S, \tag{5}$$

where f_q is the monopolar body force per unit volume, $t_q^{(n)}$ is the force (monopolar) surface traction, $T_{pq}^{(n)}$ is the dipolar (double) force surface traction, and n_p is the outward unit normal to the boundary along a section inside the body or along the surface of it. Examples of the latter tractions along the surface of a 2D half-space are given in Fig. 1. Also, a dipolar body force field is omitted in the present work since this case is a rather *unrealistic* possibility. This absence of double body forces can also be quoted in Mindlin (1964) general form and, also, in Mindlin and Eshel (1968).

In view of the above, the equations of motion and the traction boundary conditions along a smooth boundary take the form

$$\partial_p(\tau_{pq} - \partial_r m_{rpq}) + f_q = \rho \ddot{u}_q \ \textit{in}\ V \tag{6}$$

$$n_p(\tau_{pq} - \partial_r m_{rpq}) - D_p(n_r m_{rpq}) + (D_j n_j) n_r n_p m_{rpq} = P_q \ \text{on}\ \textit{bdy}, \tag{7}$$

$$n_r n_p m_{rpq} = R_q \ \text{on}\ \textit{bdy}, \tag{8}$$

where V is the region (open set) occupied by the body, *bdy* denotes any boundary along a section inside the body or along the surface of it, $D_p(\,) \equiv \partial_p(\,) - n_p D(\,)$ is the surface gradient operator, $D(\,) \equiv n_r \partial_r(\,)$ is the normal gradient operator, and $P_q \equiv t_q^{(n)} + (D_r n_r) n_p T_{pq}^{(n)} - D_p T_{pq}^{(n)}$ and $R_q \equiv n_p T_{pq}^{(n)}$ are *auxiliary* force and double-force tractions, respectively.

For completeness, the kinematical boundary conditions are also stated (Georgiadis and Grentzelou, 2006)

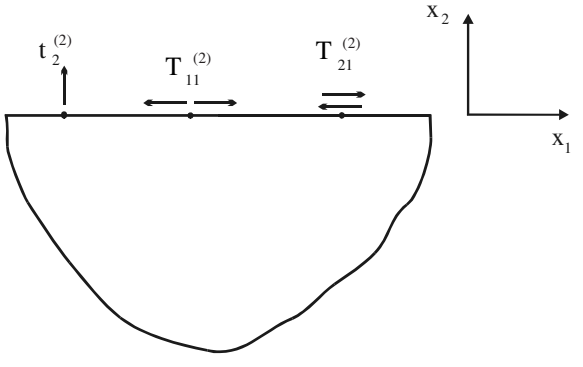

Fig. 1. Positively oriented true monopolar and dipolar tractions on the surface of a half-space.

$$u_q : \text{given on } S_u, \tag{9}$$

$$D(u_q) : \text{given on } S_u, \tag{10}$$

where S_u is the portion of the surface S of the body on which both displacements and their normal derivatives are prescribed. Of course, $S_\sigma \cup S_u = S$ and $S_\sigma \cap S_u = \emptyset$ hold true, with S_σ being the portion of the surface S of the body on which external tractions are prescribed.

In the case now of a *linear* constitutive behavior, the strain–energy density takes the following general quadratic form

$$W = (1/2)c_{pqlm}\varepsilon_{pq}\varepsilon_{lm} + (1/2)d_{rpqjlm}\kappa_{rpq}\kappa_{jlm} + f_{rpqlm}\kappa_{rpq}\varepsilon_{lm}, \tag{11}$$

where c_{pqlm}, d_{rpqjlm} and f_{rpqlm} are tensors of the material constants. The number of independent components of the tensors c_{pqlm} and d_{rpqjlm} (which are of even rank) can be reduced to yield isotropic behavior, but the tensor f_{rpqlm} (being of odd rank) inevitably results in some type of anisotropic behavior. In other words, when isotropic material behavior is to be considered, it should definitely be set $f_{rpqlm} = 0$. Therefore, in what follows, we omit the latter term and consider the following material response

$$W = (1/2)c_{pqlm}\varepsilon_{pq}\varepsilon_{lm} + (1/2)d_{rpqjlm}\kappa_{rpq}\kappa_{jlm}, \tag{12}$$

$$\tau_{pq} = c_{pqlm}\varepsilon_{lm}, \quad m_{rpq} = d_{rpqjlm}\kappa_{jlm}, \tag{13a, b}$$

where the symmetries $c_{pqlm} = c_{lmpq} = c_{qplm} = c_{pqml}$ and $d_{rpqjlm} = d_{jlmrpq} = d_{prqjlm} = d_{rpqljm}$ prevail (due to the symmetries of the stresses and the kinematical variables). In the case of non-homogeneous behavior, (c_{pqlm}, d_{rpqjlm}) can be considered as continuously differentiable functions of position $\mathbf{x}$. On the other hand, the positive definiteness of W sets restrictions on the range of values of the material constants. Inequalities of this type are given, e.g., in Georgiadis et al. (2004).

Finally, we record the constitutive relations for an *isotropic* linear gradient material (Mindlin, 1964) – a case in which our results will frequently be specialized

$$\tau_{pq} = \lambda\delta_{pq}\varepsilon_{jj} + 2\mu\varepsilon_{pq}, \tag{14a}$$

$$m_{rpq} = \frac{1}{2}d_1(\kappa_{jjr}\delta_{pq} + 2\kappa_{qjj}\delta_{rp} + \kappa_{jjp}\delta_{qr}) + d_2(\kappa_{rjj}\delta_{pq} + \kappa_{pjj}\delta_{rq}) + 2d_3\kappa_{jjq}\delta_{rp} + 2d_4\kappa_{rpq} + d_5(\kappa_{qpr} + \kappa_{qrp}), \tag{14b}$$

where λ and μ are the standard Lamé constants, $d_\beta(\beta = 1, \ldots 5)$ are the additional material constants, and δ_{pq} is the Kronecker delta.

3. Balance laws

As discussed in the Introduction, the balance laws will be obtained through the application of the *grad*, *curl* and *div* operators on the Hamiltonian of the system. We generally consider a non-homogeneous, anisotropic, elastic solid. We also assume, in the general case, inertia effects and the presence of monopolar body forces. The solid is governed by the gradient theory. In such a body, a bounded regular closed region V is considered and this region is enclosed by a surface S whose unit outward normal vector is $\mathbf{n}$. The region is simply connected and free of singularities. In accord with the discussion in Section 2, the Hamiltonian density of the system is

$$E \equiv T + W = \frac{1}{2}\rho\dot{u}_i\dot{u}_i + W(\varepsilon_{ij}, \kappa_{lij}), \tag{15}$$

in the case of general non-linear constitutive response, whereas

$$E = \frac{1}{2}\rho\dot{u}_i\dot{u}_i + \frac{1}{2}c_{ijlm}\varepsilon_{ij}\varepsilon_{lm} + \frac{1}{2}d_{lijspq}\kappa_{lij}\kappa_{spq}, \tag{16}$$

in the case of linear constitutive response. In the above equations, $u_i \equiv u_i(\mathbf{x}, t)$, $\rho \equiv \rho(\mathbf{x})$, $c_{ijlm} \equiv c_{ijlm}(\mathbf{x})$, $d_{lijspq} \equiv d_{lijspq}(\mathbf{x})$.

3.1. Gradient

The first balance law will be derived through the application of the gradient operator on the Hamiltonian. We consider a non-linear constitutive response and write

$$\nabla E \equiv E_{,k} = \frac{\partial E}{\partial \dot{u}_i}\frac{\partial \dot{u}_i}{\partial x_k} + \frac{\partial E}{\partial \varepsilon_{ij}}\frac{\partial \varepsilon_{ij}}{\partial x_k} + \frac{\partial E}{\partial \kappa_{lij}}\frac{\partial \kappa_{lij}}{\partial x_k} + \left(\frac{\partial E}{\partial x_k}\right)_{\text{expl.}} \Rightarrow$$
$$E_{,i}\delta_{ik} - \tau_{ij}u_{j,ik} - m_{lij}u_{j,ilk} - \rho\dot{u}_i\dot{u}_{i,k} = (E_{,k})_{\text{expl.}}, \tag{17}$$

where the explicit derivative of E with respect to x_k has its standard definition

$$\left(\frac{\partial E}{\partial x_k}\right)_{\text{expl.}} \equiv \frac{\partial}{\partial x_k}E(u_i, \dot{u}_i, \varepsilon_{ij}, \kappa_{lij}, x_l)\Big|_{u_i, \dot{u}_i, \varepsilon_{ij}, \kappa_{lij}\ \text{const.},\ x_l\ \text{const.},\ l \neq k}, \tag{18}$$

i.e. it is obtained by considering all other variables of E, apart from x_k, as constants.

Further, through the introduction of divergence of both the monopolar and the dipolar stresses, use of the equations of motion in (6), and an appropriate grouping of terms, (17) assumes the form

$$[E\delta_{ik} - (\tau_{ij} - m_{lij,l})u_{j,k} - m_{ilj}u_{j,lk}]_{,i} = (E_{,k})_{\text{expl.}} + \rho\dot{u}_i\dot{u}_{i,k} - \rho\ddot{u}_j u_{j,k} + f_j u_{j,k}, \tag{19}$$

which is the *differential* form of the *first* balance law obtained here. The RHS of (19) pertains to material inhomogeneity, inertia effects and body forces.

An *integral* form of (19) can be obtained through the application of the Green-Gauss theorem (divergence theorem)

$$\int_S [En_k - n_i(\tau_{ij} - m_{lij,l})u_{j,k} - n_i m_{lij}u_{j,lk}]\mathrm{d}S - \int_V [(E_{,k})_{\text{expl.}} + \rho\dot{u}_i\dot{u}_{i,k} - \rho\ddot{u}_i u_{i,k} + f_i u_{i,k}]\mathrm{d}V = 0. \tag{20}$$

This expression can also be written in terms of the auxiliary tractions (P_i, R_i). Indeed, by splitting the derivative of the displacement into a normal and a tangential part, (20) can be written as

$$\int_S [En_k - n_i(\tau_{ij} - m_{lij,l})u_{j,k} - n_i m_{lij}D_l(u_{j,k}) - n_l n_i m_{lij}D(u_{j,k})]\mathrm{d}S$$
$$- \int_V [(E_{,k})_{\text{expl.}} + \rho\dot{u}_i\dot{u}_{i,k} - \rho\ddot{u}_i u_{i,k} + f_i u_{i,k}]\mathrm{d}V = 0 \Rightarrow$$
$$\int_S [En_k - n_i(\tau_{ij} - m_{lij,l})u_{j,k} - (D_s n_s)n_l n_i m_{lij}u_{j,k} + D_l(n_i m_{lij})u_{j,k} - n_l n_i m_{lij}D(u_{j,k})]\mathrm{d}S$$
$$- \int_V [(E_{,k})_{\text{expl.}} + \rho\dot{u}_i\dot{u}_{i,k} - \rho\ddot{u}_i u_{i,k} + f_i u_{i,k}]\mathrm{d}V = 0, \tag{21}$$

where the following relation is employed

$$\int_S D_i(n_j T_{jabc\ldots})\mathrm{d}S = \int_S (D_s n_s)n_i n_j T_{jabc\ldots}\mathrm{d}S, \tag{22}$$

which holds for any smooth surface. $T_{jabc\ldots}$ stands for a Cartesian tensor of any rank. A proof of the latter statement is straightforward and can be found, e.g., in the treatise by Jaunzemis, 1967. Suffice it to say that one starts with the relation $n_k e_{krq}\partial_r(e_{lqp}n_p n_j T_{abc\ldots}) = D_l(n_j T_{abc\ldots}) - n_l n_j T_{abc\ldots}(D_k n_k)$ and then uses the Green-Gauss theorem.

Next, by utilizing the traction boundary conditions stated in Eqs. (7) and (8), we obtain

$$\int_S [En_k - P_i u_{i,k} - R_i D(u_{i,k})]\mathrm{d}S - \int_V [(E_{,k})_{expl.} + \rho\dot{u}_i\dot{u}_{i,k} - \rho\ddot{u}_i u_{i,k} + f_i u_{i,k}]\mathrm{d}V = 0. \tag{23}$$

Both relations Eqs. (20) and (23) are integral representations of the same balance law, i.e. the one stated in (19). In the absence of body forces and inhomogeneity, and in the quasi-static case (no inertial effects), (19) degenerates into the following *conservation law*

$$[W\delta_{ik} - (\tau_{ij} - m_{lij,l})u_{j,k} - m_{lij}u_{j,lk}]_{,i} = 0, \tag{24}$$

the corresponding integral form of which is

$$\int_S [Wn_k - n_i(\tau_{ij} - m_{lij,l})u_{j,k} - n_i m_{lij}u_{j,lk}]\mathrm{d}S = 0, \tag{25}$$

written also under the form (using the auxiliary tractions)

$$\int_S [Wn_k - P_i u_{i,k} - R_i D(u_{i,k})]\mathrm{d}S = 0. \tag{26}$$

The above conservation law is in agreement with the one obtained by Georgiadis and Grentzelou (2006) in the 2D case. In that case, the LHS of (26), with S being a piecewise smooth simple curve surrounding the crack tip, is the J-integral in the context of gradient elasticity.

3.2. Rotation

The second balance law is derived through the application of the rotation (*curl*) operator to the product of the Hamiltonian function and the coordinates

$$\nabla \times (E\mathbf{x}) \equiv e_{kij}(Ex_j)_{,i} = e_{kij}\left[\frac{\partial(Ex_j)}{\partial \dot{u}_n}\frac{\partial \dot{u}_n}{\partial x_i} + \frac{\partial(Ex_j)}{\partial \varepsilon_{mn}}\frac{\partial \varepsilon_{mn}}{\partial x_i} + \frac{\partial(Ex_j)}{\partial \kappa_{lmn}}\frac{\partial \kappa_{lmn}}{\partial x_i} + \left(\frac{\partial(Ex_j)}{\partial x_i}\right)_{\text{expl.}}\right] \Rightarrow$$
$$e_{kij}(Ex_j)_{,i} = e_{kij}[\rho \dot{u}_n \dot{u}_{n,i}x_j + \tau_{mn}u_{n,mi}x_j + m_{lmn}u_{n,mli}x_j + (E_{,i})_{\text{expl.}}x_j]. \tag{27}$$

Now, if the divergence of the monopolar and the dipolar stress tensors is introduced, (27) becomes

$$\begin{aligned} &e_{kij}\{(Ex_j)_{,i} - [(\tau_{mm} - m_{lmn,l})u_{n,i}x_j]_{,m} + (\tau_{mn} - m_{lmn,l})_{,m}u_{n,i}x_j + \tau_{jn}u_{n,i} - (m_{lmn}u_{n,li}x_j)_{,m} - m_{ljn,l}u_{n,i} \\ &\quad + m_{ljn}u_{n,li}\} \\ &\quad = e_{kij}[(E_{,i})_{\text{expl.}}x_j + \rho \dot{u}_n \dot{u}_{n,i}x_j]. \end{aligned} \tag{28}$$

Next, by employing the equations of motion in (6) and adding and subtracting the term $e_{kij}(\tau_{ni} - m_{lni,l})u_{j,n}$ to the LHS of Eq. (28), we obtain after proper re-arrangement of terms the following relation

$$\begin{aligned} &\{e_{kmj}Ex_j - e_{kij}[(\tau_{mn} - m_{lmn,l})u_{n,i}x_j - (\tau_{im} - m_{lmi,l})u_j + m_{lmn}u_{n,li}x_j + m_{jmn}u_{n,i} - m_{mni}u_{j,n}]\}_{,m} \\ &\quad = e_{kij}[(E_{,i})_{\text{expl.}}x_j + \rho \dot{u}_n \dot{u}_{n,i}x_j - \rho \ddot{u}_n u_{n,i}x_j + \rho \ddot{u}_i u_j + f_n u_{n,i}x_j - f_i u_j - (\tau_{jn}u_{n,i} - \tau_{in}u_{j,n} + 2m_{ljn}u_{n,il} \\ &\qquad - m_{lni}u_{j,nl})], \end{aligned} \tag{29}$$

which is the differential form of the *second* balance law obtained here. The RHS of Eq. (29) contains terms stemming from inhomogeneity, inertia, body forces and anisotropy. In particular, the term in round parentheses in the RHS of (29) is treated for linear and isotropic material behavior in our Appendix.

An integral form of the second balance law can be obtained through the application of the divergence theorem in (29). We obtain

$$\begin{aligned} &\int_S e_{kij}[n_i Ex_j - n_m(\tau_{mn} - m_{lmn,l})u_{n,i}x_j + n_m(\tau_{im} - m_{lmi,l})u_j - n_m m_{lmn}u_{n,li}x_j - n_m m_{jmn}u_{n,i} + n_m m_{mni}u_{j,n}]\mathrm{d}S \\ &\quad + \int_V e_{kij}(\tau_{jn}u_{n,i} - \tau_{in}u_{j,n} + 2m_{ljn}u_{n,il} - m_{lni}u_{j,nl})\mathrm{d}V - \int_V e_{kij}[(E_{,i})_{\text{expl.}}x_j + \rho \dot{u}_n \dot{u}_{n,i}x_j - \rho \ddot{u}_n u_{n,i}x_j + \rho \ddot{u}_i u_j \\ &\quad + f_n u_{n,i}x_j - f_i u_j]\mathrm{d}V \\ &\quad = 0. \end{aligned} \tag{30}$$

This can also be written alternatively in terms of auxiliary tractions. By splitting up the derivative of the displacement into normal and tangential parts and by using the boundary conditions stated in (7) and (8), we get after some algebra

$$\int_S e_{kij}[n_i E x_j - P_n u_{n,i} x_j + P_i u_j - R_n D(u_{n,i}) x_j + R_i D(u_j) - R_n D_i(u_n) n_j]\mathrm{d}S + \int_V e_{kij}(\tau_{jn} u_{n,i} - \tau_{in} u_{j,n} + 2m_{ljn} u_{n,il} - m_{lni} u_{j,nl})\mathrm{d}V - \int_V e_{kij}[(E_{,i})_{\text{expl.}} x_j + \rho \dot{u}_n \dot{u}_{n,i} x_j - \rho \ddot{u}_n u_{n,i} x_j + \rho \ddot{u}_i u_j + f_n u_{n,i} x_j - f_i u_j]\mathrm{d}V = 0. \tag{31}$$

Now, we restrict attention to the case of a linear, isotropic, homogeneous, grade 2, elastic material with no body forces and no inertia effects. In such a case, (29) yields the following conservation law (the result in Appendix was utilized)

$$\{e_{kmj} W x_j - e_{kij}[(\tau_{mn} - m_{lmn,l}) u_{n,i} x_j + (\tau_{im} - m_{lmi,l}) u_j - m_{lmn} u_{n,li} x_j - m_{jmn} u_{n,i} + m_{mni} u_{j,n}]\}_{,m} = 0. \tag{32}$$

The above conservation law is written by using surface integrals under the following two alternative forms. The first is written in terms of monopolar and dipolar stresses

$$\int_S e_{kij}[n_i W x_j - n_m(\tau_{mn} - m_{lmn,l}) u_{n,i} x_j + n_m(\tau_{im} - m_{lmi,l}) u_j]\mathrm{d}S - \int_S e_{kij}[n_m m_{lmn} u_{n,li} x_j + n_m m_{jmn} u_{n,i} - n_m m_{mni} u_{j,n}]\mathrm{d}S = 0, \tag{33}$$

and the second in terms of auxiliary tractions

$$\int_S e_{kij}[n_i W x_j - P_n u_{n,i} x_j + P_i u_j - R_n D(u_{n,i}) x_j + R_i D(u_j) - R_n D_i(u_n) n_j]\mathrm{d}S = 0. \tag{34}$$

3.3. Divergence

The third balance law is obtained by taking the divergence of the product of the Hamiltonian function of the system and the coordinates

$$\begin{aligned} &\nabla \cdot (E\mathbf{x}) \equiv (E x_k)_{,k} = \frac{\partial (E x_k)}{\partial \dot{u}_i} \frac{\partial \dot{u}_i}{\partial x_k} + \frac{\partial (E x_k)}{\partial \varepsilon_{ij}} \frac{\partial \varepsilon_{ij}}{\partial x_k} + \frac{\partial (E x_k)}{\partial \kappa_{lij}} \frac{\partial \kappa_{lij}}{\partial x_k} + \left(\frac{\partial (E x_k)}{\partial x_k}\right)_{\text{expl.}} \Rightarrow \\ &(E x_k)_{,k} = \rho \dot{u}_i \dot{u}_{i,k} x_k + \tau_{ij} u_{j,ik} x_k + m_{lij} u_{j,ilk} x_k + (E_{,k})_{\text{expl.}} x_k + E x_{k,k} \Rightarrow \\ &(E x_k)_{,k} = \rho \dot{u}_i \dot{u}_{i,k} x_k + \tau_{ij} u_{j,ik} x_k + m_{lij} u_{j,ilk} x_k + (E_{,k})_{\text{expl.}} x_k + \alpha E, \end{aligned} \tag{35}$$

where α is equal to the dimension of space ($\alpha = 2$ in 2D problems, and $\alpha = 3$ in 3D problems). Further, by introducing the divergence of stress tensors and re-arranging terms properly, we obtain

$$\begin{aligned} &(E x_k)_{,k} - (\tau_{ij} u_{j,k} x_k)_{,i} + \tau_{ij,i} u_{j,k} x_k + \tau_{ij} u_{j,i} - (m_{lij} u_{j,ik} x_k)_{,l} + (m_{lij,l} u_{j,k} x_k)_{,i} - m_{lij,li} u_{j,k} x_k - m_{lij,l} u_{j,i} \\ &- \frac{\alpha - 2}{2}(\tau_{ij} u_j)_{,i} + \frac{\alpha - 2}{2} \tau_{ij,i} u_j - \frac{\alpha - 2}{2}(m_{lij} u_{j,i})_{,l} + \frac{\alpha - 2}{2} m_{lij,l} u_{j,i} \\ &= (E_{,k})_{\text{expl.}} x_k + \rho \dot{u}_i \dot{u}_{i,k} x_k + \frac{\alpha}{2} \rho \dot{u}_i \dot{u}_i. \end{aligned} \tag{36}$$

Finally, by employing the equations of motion, (36) assumes the form

$$\begin{aligned} &\left[E x_i - (\tau_{ij} - m_{lij,l}) u_{j,k} x_k - m_{ilj} u_{j,lk} x_k - \frac{\alpha - 2}{2}(\tau_{ij} - m_{lij,l}) u_j - \frac{\alpha}{2} m_{ilj} u_{j,l}\right]_{,i} \\ &= (E_{,k})_{\text{expl.}} x_k + \rho \dot{u}_i \dot{u}_{i,k} x_k - \rho \ddot{u}_i u_{i,k} x_k + \frac{\alpha}{2} \rho \dot{u}_i \dot{u}_i - \frac{\alpha - 2}{2} \rho \ddot{u}_i u_i + f_i u_{i,k} x_k + \frac{\alpha - 2}{2} f_i u_i + m_{lij} u_{j,il}, \end{aligned} \tag{37}$$

which is the differential form of the *third* balance law obtained here.

In order to have the integral form of this balance law, we apply the divergence theorem and get

$$\int_S n_i\left[Ex_i-(\tau_{ij}-m_{lij,l})u_{j,k}x_k-m_{ilj}u_{j,lk}x_k-\frac{\alpha-2}{2}(\tau_{ij}-m_{lij,l})u_j-\frac{\alpha}{2}m_{ilj}u_{j,l}\right]\mathrm{d}S$$
$$-\int_V\left[(E_{,k})_{\text{expl.}}x_k+\rho\dot{u}_i\dot{u}_{i,k}x_k+\frac{\alpha}{2}\rho\dot{u}_i\dot{u}_i-\rho\ddot{u}_iu_{i,k}x_k-\frac{\alpha-2}{2}\rho\ddot{u}_iu_i+f_iu_{i,k}x_k+\frac{\alpha-2}{2}f_iu_i\right]\mathrm{d}V$$
$$-\int_V m_{lij}u_{j,il}\mathrm{d}V=0. \tag{38}$$

Again, this can also be written in terms of auxiliary tractions. Through a process similar to the previous cases, we finally have

$$\int_S\left[En_kx_k-P_iu_{i,k}x_k-\frac{\alpha-2}{2}P_iu_i-R_iD(u_{i,k})x_k-\frac{\alpha}{2}R_iD(u_i)\right]\mathrm{d}S$$
$$-\int_V\left[(E_{,k})_{\text{expl.}}x_k+\rho\dot{u}_i\dot{u}_{i,k}x_k+\frac{\alpha}{2}\rho\dot{u}_i\dot{u}_i-\rho\ddot{u}_iu_{i,k}x_k-\frac{\alpha-2}{2}\rho\ddot{u}_iu_i+f_iu_{i,k}x_k+\frac{\alpha-2}{2}f_iu_i\right]\mathrm{d}V$$
$$-\int_V m_{lij}u_{j,il}\mathrm{d}V=0. \tag{39}$$

Notice that the above balance law *never* leads to a conservation law, in contrast to the two previous occasions in (26) and (34). Indeed, even for a homogeneous material in the absence of body forces and inertia effects, there still remains the term $m_{lij}u_{j,il}$ (third integral) in the LHS of Eq. (39).

4. Energy release rates for cracks

In this Section, we will derive the expressions for the energy release rates associated with certain crack (defect) *motions*, namely (i) rigid translation, (ii) rigid rotation, and (iii) self-similar expansion. In the first case, all points on the crack surfaces are considered to move with velocity $\mathbf{v}=v_k\hat{\mathbf{e}}_k$, where v_k is constant. In the second case, all points on the crack surfaces are considered to move with an angular velocity $v_k=-e_{3kl}x_l\,\omega$ about the x_3-axis, where ω is a positive constant. In the third case, all points on the crack surfaces are considered to move with velocity $v_k=ax_k$ where a is a positive constant. Comparing the results that will be obtained in this Section with the ones in the Section 3, one gains insight about the physical meaning of the conservation laws established previously. A 2D plane-strain state is assumed but the extension to a 3D situation is straightforward. Our analysis is in the spirit of the one by Eischen and Herrmann, 1987, who treated the standard elasticity case.

The case of a grade 2, elastic material is considered. In general, non-linear constitutive equations and inertia, inhomogeneity and anisotropy effects are included in the analysis, as well as the action of a body-force field. Our analysis applies also to rapid crack propagation. We consider a body occupying the plane domain Ω with its outer boundary being a smooth curve denoted by Γ_0 (see Fig. 2). The body contains a single, internal, through-the-thickness, straight crack with stress-free faces. The crack is denoted by L. The inner boundary of Ω is made up of two straight lines along the crack faces Γ_c, and two circles Γ_ε, the centre of which is situated on each of the crack tips and their radius is $\varepsilon\to 0$.

During any of the aforementioned crack motions, an energy rate balance condition must hold (Freund, 1972). According to this condition, at any instant of time t, the rate of work of tractions on Γ_0 and of the body-force field on Ω is equal to the rate of increase of the total energy of the body plus the rate of energy absorption by the moving crack surfaces, i.e.

$$P+B=\dot{K}+\dot{U}+F, \tag{40}$$

where

$$P=\int_{\Gamma_0}[P_i\dot{u}_i+R_iD(\dot{u}_i)]\mathrm{d}\Gamma, \tag{41}$$

is the rate of work of the tractions on Γ_0,

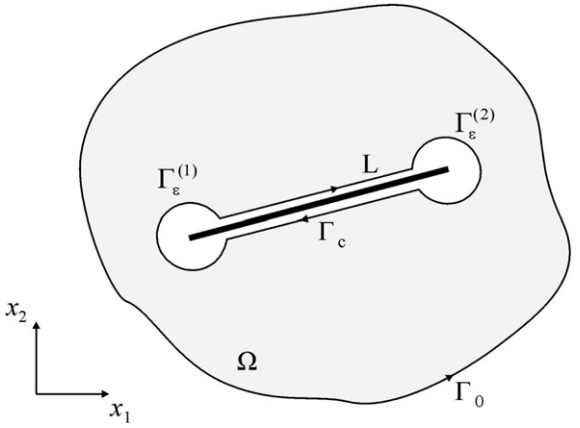

Fig. 2. A plane domain containing a finite-length crack with stress-free faces.

$$B = \lim_{\Gamma_\varepsilon \to 0} \int_{\Omega(t)} f_i \dot{u}_i \mathrm{d}\Omega, \tag{42}$$

is the rate of work of the body forces in Ω,

$$K = \lim_{\Gamma_\varepsilon \to 0} \int_{\Omega(t)} \frac{1}{2} \rho \dot{u}_i \dot{u}_i \mathrm{d}\Omega, \tag{43}$$

is the total kinetic energy in Ω, and

$$U = \lim_{\Gamma_\varepsilon \to 0} \int_{\Omega(t)} \left[\frac{1}{2} \tau_{ij} u_{j,i} + \frac{1}{2} m_{lij} u_{j,il} \right] \mathrm{d}\Omega, \tag{44}$$

is the total strain energy of the body stored in Ω. Also, in (40), F is the rate of energy absorbed into the crack tips and the crack surfaces. This rate of energy is measured as a change in energy per unit thickness, per unit time.

The position of the surfaces Γ_ε and Γ_c is time-dependent, since they move with the crack. Therefore, for the calculation of $\dot{K}$ and $\dot{U}$ we have to employ the Reynolds transport theorem (see e.g. Jaunzemis, 1967). The results of this calculation are

$$\dot{K} = \lim_{\Gamma_\varepsilon \to 0} \left\{ \int_\Omega \rho \ddot{u}_i \dot{u}_i \mathrm{d}\Omega + \int_{\Gamma_\varepsilon + \Gamma_c} \frac{1}{2} \rho \dot{u}_i \dot{u}_i v_k n_k \mathrm{d}\Gamma \right\}, \tag{45}$$

for the rate of kinetic energy and

$$\dot{U} = \lim_{\Gamma_\varepsilon \to 0} \left\{ \int_\Omega \left(\tau_{ij} \dot{u}_{j,i} + m_{lij} \dot{u}_{j,il} \right) \mathrm{d}\Omega + \int_{\Gamma_\varepsilon + \Gamma_c} \left(\frac{1}{2} \tau_{ij} u_{j,i} + \frac{1}{2} m_{lij} u_{j,il} \right) v_k n_k \mathrm{d}\Gamma \right\}, \tag{46}$$

for the rate of the strain energy.

Given the above definitions and calculations, we derive from relation (40) the expression for the energy absorption rate. Employing the equations of motion in (6) and also (41), (42), (45) and (46), Eq. (40) can be written as

$$F = \int_{\Gamma_0} [P_i \dot{u}_i + R_i D(\dot{u}_i)] \mathrm{d}\Gamma - \lim_{\Gamma_\varepsilon \to 0} \left\{ \int_\Omega [(\tau_{ij} - m_{lij,l})_{,i} \dot{u}_j + (\tau_{ij} \dot{u}_{j,i} + m_{lij} \dot{u}_{j,il})] \mathrm{d}\Omega \right\} - \lim_{\Gamma_\varepsilon \to 0} \int_{\Gamma_\varepsilon + \Gamma_c} E v_k n_k \mathrm{d}\Gamma, \tag{47}$$

which holds in the general case of a grade 2 elastic material with non-linear constitutive relations and including the effects of inertia, inhomogeneity, anisotropy and body forces. The only restriction in deriving (47) is the assumption of small displacements, strains and rotations. Then, in light of the divergence theorem and the traction boundary conditions in (7) and (8), Eq. (47) takes finally the form

$$F = -\lim_{\Gamma_\varepsilon \to 0} \int_{\Gamma_\varepsilon + \Gamma_c} [E v_k n_k + n_i(\tau_{ij} - m_{lij,l})\dot{u}_j + n_l m_{lij}\dot{u}_{j,i}]\mathrm{d}\Gamma. \tag{48}$$

Further, this energy rate can be split up into two parts, the first pertaining to the energy absorbed at the crack tips and the second to the energy absorbed along the crack surfaces

$$F = F_{\text{tip}} + F_{\text{sur}}, \tag{49}$$

where

$$F_{\text{tip}} = -\lim_{\Gamma_\varepsilon \to 0} \int_{\Gamma_\varepsilon} [E v_k n_k + n_i(\tau_{ij} - m_{lij,l})\dot{u}_j + n_l m_{lij}\dot{u}_{j,i}]\mathrm{d}\Gamma, \tag{50}$$

and

$$F_{\text{sur}} = -\lim_{\Gamma_\varepsilon \to 0} \int_{\Gamma_c} E v_k n_k \mathrm{d}\Gamma. \tag{51}$$

Note that the crack surfaces are considered traction free and that $n_k^+ = -n_k^-$ on Γ_c. Also, near the tip of an extending crack the field quantities obey the following relation (Nakamura et al., 1985)

$$\frac{\partial()}{\partial t} = -v_k \frac{\partial()}{\partial x_k}, \tag{52}$$

which is valid not only for a steady state but also for *transient* states as well.

Therefore, the energy absorption rates on the surfaces of the crack and at the crack tips are given as

$$F_{\text{tip}} = -\lim_{\Gamma_\varepsilon \to 0} \int_{\Gamma_\varepsilon} [E v_k n_k - n_i(\tau_{ij} - m_{lij,l})u_{j,k} v_k - n_l m_{lij} u_{j,ik} v_k]\mathrm{d}\Gamma, \tag{53}$$

$$F_{\text{sur}} = \lim_{\Gamma_\varepsilon \to 0} \int_{\Gamma_c} [E^- - E^+] v_k n_k^+ \mathrm{d}\Gamma. \tag{54}$$

Next, we proceed in evaluating the energy release rates for each one of the different crack (defect) motions.

4.1. Rigid translation

In the case of a conceptual rigid translation, all points on the crack surfaces are considered to move with velocity $\mathbf{v} = v_k \hat{\mathbf{e}}_k$, where v_k is constant. The energy release rate measured as a change in energy per unit translation, per unit thickness of the body, is denoted as G_k^T and is defined through

$$G_k^T \equiv \frac{F_{\text{tip}}}{v_k} = -\lim_{\Gamma_\varepsilon \to 0} \int_{\Gamma_\varepsilon} [E n_k - n_i(\tau_{ij} - m_{lij,l})u_{j,k} - n_l m_{lij} u_{j,ik}]\mathrm{d}\Gamma. \tag{55}$$

By using the divergence theorem, this can be written in terms of line integrals along the paths Γ_0 and Γ_c and a domain integral over Ω. We introduce the symbol J_k and write

$$J_k \equiv G_k^T = \int_{\Gamma_0} [E n_k - n_i(\tau_{ij} - m_{lij,l})u_{j,k} - n_l m_{lij} u_{j,ik}]\mathrm{d}\Gamma + \int_{\Gamma_c} E n_k \mathrm{d}\Gamma - \int_\Omega \{E_{,k} - [(\tau_{ij} m_{lij,l})u_{j,k}]_{,i} - (m_{lij} u_{j,ik})_{,l}\}\mathrm{d}\Omega. \tag{56}$$

Next, by employing the explicit derivative of the Hamiltonian function E and the traction boundary conditions, we obtain the following expression

$$J_k = \int_{\Gamma_0} [E n_k - P_i u_{i,k} - R_i D(u_{i,k})]\mathrm{d}\Gamma + \int_{\Gamma_c} [E^- - E^+] n_k^+ \mathrm{d}\Gamma - \int_\Omega [\rho \dot{u}_i \dot{u}_{i,k} - \rho \ddot{u}_i u_{i,k} + f_i u_{i,k} + (E_{,k})_{\text{expl.}}]\mathrm{d}\Omega. \tag{57}$$

This expression may be thought of as an extension, including gradient effects, of the respective integral derived by Eischen and Herrmann (1987).

4.2. Rigid rotation

For rigid rotation, all points on the crack surfaces are considered to move with an angular velocity $v_k = -e_{3kl}x_l\omega$ about the x_3-axis, where ω is a positive constant. The energy release rate measured as a change in energy per unit angular rotation, per unit thickness of the body, is denoted as G^R and is defined through

$$G^R \equiv \frac{F_{\text{tip}}}{\omega} = \lim_{\Gamma_\varepsilon \to 0} \int_{\Gamma_\varepsilon} e_{3kr}[En_k x_r - n_i(\tau_{ij} - m_{lij,l})u_{j,k}x_r - n_l m_{lij}u_{j,ik}x_r]\mathrm{d}\Gamma. \tag{58}$$

Again, by using the divergence theorem, this quantity can be expressed in terms of line integrals along the paths Γ_0 and Γ_c and a domain integral over Ω. Introducing the symbol L_3, we write

$$\begin{aligned} L_3 \equiv -G^R = &\int_{\Gamma_0} e_{3kr}[En_k x_r - n_i(\tau_{ij} - m_{lij,l})u_{j,k}x_r - n_l m_{lij}u_{j,ik}x_r]\mathrm{d}\Gamma \\ &+ \int_{\Gamma_c} e_{3kr}[En_k x_r]\mathrm{d}\Gamma - \int_\Omega e_{3kr}\{(Ex_r)_{,k} - [(\tau_{ij} - m_{lij,l})u_{j,k}x_r]_{,i} - (m_{lij}u_{j,ik}x_r)_{,l}\}\mathrm{d}\Omega, \end{aligned} \tag{59}$$

or, equivalently,

$$\begin{aligned} L_3 = &\int_{\Gamma_0} e_{3kr}[En_k x_r - n_i(\tau_{ij} - m_{lij,l})u_{j,k}x_r - n_l m_{lij}u_{j,ik}x_r]\mathrm{d}\Gamma + \int_{\Gamma_c} e_{3kr}En_k x_r\mathrm{d}\Gamma - \int_\Omega e_{3kr}\{(m_{lri}u_{i,k})_{,l} \\ &- (m_{lik}u_{r,i})_{,l} - [(\tau_{ik} - m_{lik,l})u_r]_{,i}\}\mathrm{d}\Omega - \int_\Omega e_{3kr}[(E_{,k})_{\text{expl.}}x_r + \rho\dot{u}_i\dot{u}_{i,k}x_r - \rho\ddot{u}_i u_{i,k}x_r + \rho\ddot{u}_k u_r + f_i u_{i,k}x_r \\ &- f_k u_r]\mathrm{d}\Omega - \int_\Omega e_{3kr}[\tau_{ik}u_{r,i} - \tau_{ri}u_{i,k} + 2m_{lri}u_{i,kl} - m_{lik}u_{r,il}]\mathrm{d}\Omega. \end{aligned} \tag{60}$$

Finally, in terms of auxiliary tractions, the above relation assumes the form

$$\begin{aligned} L_3 = &\int_{\Gamma_0} e_{3kr}[En_k x_r - P_i u_{i,k}x_r - R_i D(u_{i,k})x_r - R_i u_{i,k}Dx_r + n_l m_{lri}u_{i,k}]\mathrm{d}\Gamma + \int_{\Gamma_c} e_{3kr}En_k x_r\mathrm{d}\Gamma \\ &- \int_\Omega e_{3kr}\{(m_{lri}u_{i,k})_{,l} - (m_{lik}u_{r,i})_{,l} - [(\tau_{ik} - m_{lik,l})u_r]_{,i}\}\mathrm{d}\Omega - \int_\Omega e_{3kr}[(E_{,k})_{\text{expl.}}x_r + \rho\dot{u}_i\dot{u}_{i,k}x_r \\ &- \rho\ddot{u}_i u_{i,k}x_r + \rho\ddot{u}_k u_r + f_i u_{i,k}x_r - f_k u_r]\mathrm{d}\Omega - \int_\Omega e_{3kr}[\tau_{ik}u_{r,i} - \tau_{ri}u_{i,k} + 2m_{lri}u_{i,kl} - m_{lik}u_{r,il}]\mathrm{d}\Omega. \end{aligned} \tag{61}$$

4.3. Self-similar expansion

In the case of self-similar expansion, all points on the crack surfaces are considered to move with velocity $v_k = ax_k$ with a being a positive constant. The energy release rate measured as change in energy per unit thickness is denoted as G^E and is defined through

$$G^E \equiv \frac{F_{\text{tip}}}{a} = -\lim_{\Gamma_\varepsilon \to 0} \int_{\Gamma_\varepsilon} [Ex_k n_k - n_i(\tau_{ij} - m_{lij,l})u_{j,k}x_k - n_l m_{lij}u_{j,ik}x_k]\mathrm{d}\Gamma. \tag{62}$$

This can be written in terms of line integrals along the paths Γ_0 and Γ_c and a domain integral over Ω using the divergence theorem. We introduce the symbol M and obtain

$$\begin{aligned} M \equiv G^E = &\int_{\Gamma_0} [Ex_k n_k - n_i(\tau_{ij} - m_{lij,l})u_{j,k}x_k - n_l m_{lij}u_{j,ik}x_k]\mathrm{d}\Gamma + \int_{\Gamma_c} [E^- - E^+]n_k^+ x_k\mathrm{d}\Gamma \\ &- \int_\Omega \{(Ex_k)_{,k} - [(\tau_{ij} - m_{lij,l})u_{j,k}x_k]_{,i} - (m_{lij}u_{j,ik}x_k)_{,l}\}\mathrm{d}\Omega, \end{aligned} \tag{63}$$

or, equivalently,

$$M=\int_{\Gamma_0}[Ex_kn_k-n_i(\tau_{ij}-m_{lij,l})u_{j,k}x_k-n_lm_{lij}u_{j,ik}x_k]\mathrm{d}\Gamma+\int_{\Gamma_c}En_kx_k\mathrm{d}\Gamma$$
$$-\int_{\Omega}[(E_{,k})_{\text{expl.}}x_k+\rho\dot{u}_i\dot{u}_{i,k}x_k+\rho\dot{u}_i\dot{u}_i-\rho\ddot{u}_iu_{i,k}x_k+f_iu_{i,k}x_k]\mathrm{d}\Omega-\int_{\Omega}m_{lij,l}u_{j,i}\mathrm{d}\Omega. \tag{64}$$

Also, using auxiliary tractions, this relation takes the form

$$M=\int_{\Gamma_0}[Ex_kn_k-P_iu_{i,k}x_k-R_iD(u_{i,k})x_k-R_iD(u_i)]\mathrm{d}\Gamma+\int_{\Gamma_c}En_kx_k\mathrm{d}\Gamma-\int_{\Omega}[(E_{,k})_{\text{expl.}}x_k+\rho\dot{u}_i\dot{u}_{i,k}x_k$$
$$+\rho\dot{u}_i\dot{u}_i-\rho\ddot{u}_iu_{i,k}x_k+f_iu_{i,k}x_k]\mathrm{d}\Omega-\int_{\Omega}m_{lij}u_{j,il}\mathrm{d}\Omega. \tag{65}$$

5. Interrelation between balance laws and energy release rates

In classical elasticity, the J_k, L_k and M integrals have been associated with energy release rates (Budiansky and Rice, 1973; Eischen and Herrmann, 1987). Within the framework of dipolar gradient elasticity now, in order for us to assert that the balance laws derived in Section 3 are, in fact, the generalizations of the aforementioned integrals, we have to associate them with the integral expressions derived in Section 4. In other words, we answer below the question of whether the balance laws in (20), (30) and (38) lead to crack-tip energy release rates when the path S is taken as Γ_ε and the domain V vanishes. We restrict attention to the case of a linear, grade 2, elastic material with inhomogeneity, anisotropy and inertia effects. In particular, the restriction to linearity of constitutive equations is imposed here in order that certain domain integrals be vanished in the process of shrinking the path of integration onto the crack tips. This point is further explained below.

When the path of integration S in the expressions of the balance laws (in the two-dimensional case) is shrunk onto the crack tips, then the resulting expressions should be the energy release rates for the three individual crack motions. In this case, the domain integrals vanish (this is explained below) and Eqs. (20), (30) and (38) yield, respectively

$$I_k^{(1)}=\lim_{S\to 0}\int_S[En_k-n_i(\tau_{ij}-m_{lij,l})u_{j,k}-n_lm_{lij}u_{j,ik}]\mathrm{d}S, \tag{66}$$

$$I_3^{(2)}=\lim_{S\to 0}\int_S e_{3ij}[En_ix_j-n_m(\tau_{mn}-m_{lmn,l})u_{n,i}x_j-n_mm_{lmn}u_{n,li}x_j]\mathrm{d}S, \tag{67}$$

$$I^{(3)}=\lim_{S\to 0}\int_S[En_ix_i-n_i(\tau_{ij}-m_{lij,l})u_{j,k}x_k-n_lm_{lij}u_{j,ik}x_k]\mathrm{d}S, \tag{68}$$

Indeed, the above equations are identified with (55), (58) and (62). Therefore, if the surface of integration encloses a defect, each balance law corresponds to an energy release rate for a particular motion of that defect: translation ($I_k^{(1)}=-G_k^T$), rotation ($I_3^{(2)}=-G_3^R$), and self-similar expansion ($I^{(3)}=-G^E$). The opposite signs in these expressions are due to the fact that the paths S and Γ_ε are traversed in opposite directions.

As now for the question of whether the domain integrals in (20), (30) and (38) vanish when S is taken to be Γ_ε in the limit $\Gamma_\varepsilon\to 0$, the answer in the case of linear gradient elasticity is in the affirmative. The argument is based on the following points: (i) time and space derivatives are of the *same* order, as (52) suggests, in the crack-tip vicinity, (ii) the strain–energy density and the displacement and body-force fields should be bounded in crack-tip vicinity, in view of the uniqueness theorem for 2D crack problems (Grentzelou and Georgiadis, 2005), and (iii) the displacement behaves as $r^{3/2}$ and the dipolar stresses as $r^{-1/2}$, where r is the radial distance from the crack tip, in 2D crack problems of gradient elasticity (see e.g. Georgiadis, 2003).

In the quasi-static and homogeneous case, one defines the following conservative integral through the corresponding conservation law in (25) and (26)

$$J_k\equiv\int_S[Wn_k-n_i(\tau_{ij}-m_{lij,l})u_{j,k}-n_im_{lij}u_{j,lk}]\mathrm{d}S=\int_S[Wn_k-P_iu_{i,k}-R_iD(u_{i,k})]\mathrm{d}S, \tag{69}$$

which is the 'gradient-elasticity' generalization of the counterpart integral of standard elasticity. In addition, the integral in (69) is the generalization in *three* dimensions of the J-integral, in gradient elasticity, previously derived through an elementary procedure by Georgiadis and Grentzelou (2006).

Also, in the quasi-static, homogeneous and isotropic case, one may define the following conservative integral through the corresponding conservation law in (33) and (34)

$$\begin{aligned} L_k &\equiv \int_S e_{kij}[n_i W x_j - n_m(\tau_{mn} - m_{lmn,l})u_{n,i}x_j + n_m(\tau_{im} - m_{lmi,l})u_j]\mathrm{d}S - \int_S e_{kij}[n_m m_{lmn}u_{n,li}x_j + n_m m_{jmn}u_{n,i} \\ &\quad - n_m m_{mni}u_{j,n}]\mathrm{d}S \\ &= \int_S e_{kij}[n_i W x_j - P_n u_{n,i}x_j + P_i u_j - R_n D(u_{n,i})x_j + R_i D(u_j) - R_n D_i(u_n)n_j]\mathrm{d}S, \end{aligned} \tag{70}$$

which is the 'gradient-elasticity' generalization of the counterpart integral of standard elasticity.

Finally, note that, in view of (39), the M-integral defined in (65) is not path-independent even in the quasi-static, homogeneous case with zero body forces. This is because the existence of characteristic lengths in the material response renders the strain–energy density *non-invariant* under a self-similar scale change. A similar conclusion was also reached by Lubarda and Markenskoff (2000) in the case of couple-stress elasticity.

6. A note on the strain-gradient case

In the 'pure' strain-gradient case, Eq. (2) for the strain–energy density is considered but now with κ_{rpq} being the gradient of *only* the strain field (and not of the entire displacement-gradient field), i.e. $\kappa_{rpq} = \partial_r \varepsilon_{pq}$. This is form II in Mindlin's (1964) paper. Obviously, $\kappa_{rpq} \equiv \kappa_{rqp}$ holds. Stresses are defined as in (3a,b) and, therefore, the dipolar stress tensor exhibits the symmetry $m_{rpq} \equiv m_{rqp}$. In this form, rotation gradients are not considered and couple-stresses are absent. All governing equations pertaining to the general form and all energy considerations and results given before are also valid for form II provided that the *proper* symmetries for all tensors are followed.

7. Conclusions

The Hamiltonian of a grade 2 elastic material is subjected to the *grad* operator and the product of the Hamiltonian and the coordinates is subjected to the *curl* and *div* operators. In this way, three balance laws are obtained, which are related to the energy release rates for crack translation, crack rotation and self-similar expansion. In the quasi-static case and in the absence of inertial effects and body forces, if the material is homogeneous the first balance law is in fact a conservation law and yields a path-independent integral analogous to the $\boldsymbol{J}$ integral of classical elasticity. If in addition the material is considered to be isotropic, the second balance law becomes a conservation law as well, and a path-independent integral of the $\boldsymbol{L}$ type is obtained. It is shown, however, that a path-independent integral of the M-type does not exist in dipolar gradient elasticity.

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Appendix A

Here, we prove that for an isotropic, linear, grade 2, elastic material, the expression we define as b_k below is equal to zero

$$b_k \equiv e_{kij}[\tau_{jn}u_{n,i} - \tau_{in}u_{j,n} + 2m_{ljn}u_{n,il} - m_{lni}u_{j,nl}]. \tag{A1}$$

To this end, we replace in this expression the constitutive relations (14) and write

$$\begin{aligned}
b_k &= e_{kij}[\lambda\varepsilon_{ss}u_{j,i} + 2\mu\varepsilon_{jn}u_{n,i} - \lambda\varepsilon_{ss}u_{j,i} - 2\mu\varepsilon_{in}u_{j,n} + d_1(u_{l,ss}u_{j,il} + 2u_{s,sl}u_{l,ij} + u_{j,ss}u_{l,il}) + 2d_2(u_{s,sl}u_{j,il} + u_{s,sj}u_{l,il}) \\
&\quad + 4d_3u_{n,ss}u_{n,ij} + 4d_4u_{n,jl}u_{n,il} + 2d_5(u_{l,jn}u_{n,il} + u_{j,ln}u_{n,il}) - d_1(u_{n,ss}u_{j,in} + u_{s,si}u_{j,nn}) - 2d_2u_{s,sn}u_{j,in} \\
&\quad - 2d_3u_{i,ss}u_{j,ll} - 2d_4u_{i,nl}u_{j,nl} - 2d_5u_{l,ni}u_{j,nl}] \Rightarrow \\
b_k &= e_{kij}[2\mu(\varepsilon_{jn}u_{n,i} - \varepsilon_{in}u_{j,n}) + 2d_2u_{n,nj}u_{l,il} - 2d_3u_{i,nn}u_{j,ll} + 4d_4u_{n,il}u_{n,jl} - 2d_4u_{i,nl}u_{j,nl} + 2d_5u_{l,jn}u_{n,il}] \Rightarrow \\
b_k &= e_{kij}(2d_2u_{n,nj}u_{l,il} - 2d_3u_{i,nn}u_{j,ll} + 4d_4u_{n,il}u_{n,jl} - 2d_4u_{i,nl}u_{j,nl} + 2d_5u_{l,jn}u_{n,il}),
\end{aligned} \tag{A2}$$

where we have employed the result that the term $e_{kij}\,(\varepsilon_{jn}u_{n,i} - \varepsilon_{in}u_{j,n})$ vanishes:

$$\begin{aligned}
&e_{kij}\varepsilon_{jn}u_{n,i} - e_{kij}\varepsilon_{in}u_{j,n} = e_{kij}\varepsilon_{jn}u_{n,i} + e_{kji}\varepsilon_{in}u_{j,n} = e_{kij}\varepsilon_{jn}u_{n,i} + e_{kij}\varepsilon_{jn}u_{i,n} \\
&\quad = e_{kij}\varepsilon_{jn}(u_{n,i} + u_{i,n}) = 2e_{kij}\varepsilon_{jn}\varepsilon_{in} = 0.
\end{aligned} \tag{A3}$$

Next, we multiply both sides of (A2) with e_{kst} and employ the relation $e_{kij}e_{kst} = \delta_{is}\delta_{jt} - \delta_{it}\delta_{js}$. Then, (A2) provides

$$\begin{aligned}
e_{kst}b_k &= (\delta_{is}\delta_{jt} - \delta_{it}\delta_{js})(2d_2u_{n,nj}u_{l,il} - 2d_3u_{i,nn}u_{j,ll} + 4d_4u_{n,il}u_{n,jl} - 2d_4u_{i,nl}u_{j,nl} + 2d_5u_{l,jn}u_{n,il}) \Rightarrow \\
e_{kst}b_k &= 2d_2(u_{n,nt}u_{l,ls} - u_{n,ns}u_{l,lt}) - 2d_3(u_{s,nn}u_{t,ll} - u_{t,nn}u_{s,ll}) + 4d_4(u_{n,sl}u_{n,tl} - u_{n,tl}u_{n,sl}) \\
&\quad - 2d_4(u_{s,nl}u_{t,nl} - u_{t,nl}u_{s,nl}) + 2d_5(u_{l,tn}u_{n,sl} - u_{l,sn}u_{n,tl}) \Rightarrow e_{kst}b_k = 0 \Rightarrow b_k = 0,
\end{aligned} \tag{A4}$$

which concludes the proof of our statement.

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